

Long-range correlations of neutrinos in hadron reactions and neutrino diffraction I: Formalism

K. Ishikawa and Y. Tobita

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Department of Physics, Faculty of Science,
Hokkaido University, Sapporo 060-0810, Japan

Abstract

A finite-size correction to a probability to detect the neutrino produced in high-energy hadron reactions is studied in this series of papers I and II. The neutrino is produced as a secondary particle from a weak decay of stable particles produced in hadron reactions and detected at a distant position with a target nucleus. In I, we describe a general formalism on the finite-size corrections and particularly on those of the pion in hadron collisions. The finite-size correction has an origin in a many-body wave function at a finite time in which the kinetic energy is not a good quantum number. The state is non-uniform in space-time and reveals interference pattern that can be computed rigorously via the light-cone singularity of the involving particles. The correction depends on the mass and energy of the observed particle and becomes small for the pion. In the following paper II, the neutrino in the pion decay is shown to reveal a large finite-size correction that depends on the absolute neutrino mass in a universal manner. Thus a new method of measuring the absolute neutrino mass is suggested.

1 Introduction

Understanding neutrinos is in rapid progress these days. Neutrinos are very light and mass squared differences were given from recent flavor oscillation experiments [1, 2, 3, 4, 5, 6] using neutrinos from the sun, accelerators, reactors, and atmosphere as [7],

$$\Delta m_{21}^2 = m_2^2 - m_1^2 = (7.50 \pm 0.20) \times 10^{-5} [\text{eV}^2/c^4], \quad (1)$$

$$|\Delta m_{32}^2| = |m_3^2 - m_2^2| = (2.32_{-0.08}^{+0.13}) \times 10^{-3} [\text{eV}^2/c^4], \quad (2)$$

where m_i ($i = 1 - 3$) are mass values. Absolute masses are important but are not found from oscillation experiments. Tritium beta decays [8] have been used for determining the absolute value but an existing upper bound for an effective electron neutrino mass-squared is of the order of 2 [eV^2/c^4] and the mass is 0.3 – 1.3 [eV/c^2] from cosmological observations [9]. In these neutrino experiments, higher precision and more statistics have been achieved and will be improved more. In this series of papers, we study an interference phenomenon of neutrinos that could give informations of the absolute masses.

A neutrino interacts extremely weakly with matter so is idealistic for a test of quantum mechanics. Although its validity is certain from many tests and applications made in the past, most of them are restricted mainly to microscopic areas. Neutrinos provide a new test in macroscopic regions. For the test, the number of neutrino events must be large to have small statistical fluctuations. In electron bi-prism experiments by Tonomura et al [10], interferences of electrons of a sharp energy become visible when the total number becomes significant. Even though initial electrons are created randomly, when the number of events becomes large and a signal exceeds a statistical fluctuation, a distribution of detected electrons displays an interference pattern. Each pion decays randomly but the entire decay process is described by a quantum mechanical wave function and the distribution of the produced neutrino could display a similar interference pattern. For the interference pattern to be observed, a large number of events is necessary.

Charged current weak decays occur with an interaction Lagrangian of a product of $V - A$ currents in low energy region and a neutrino and charged lepton are produced in pair. Weak decay processes have been studied using an S-matrix of plane waves with asymptotic boundary conditions, where particles in the initial and final states are regarded as free waves without

correlations [11, 12, 13, 14, 15]. A decay rate, average life time, and various distributions of charged leptons have been computed, and perfect agreements with experiments have been obtained [16]. The charged leptons are in fact in the asymptotic space-time regions in these experiments. Probabilities to detect particles in asymptotic regions do not depend on distances between positions of the production and the detection, and are computed with the standard S-matrix of plane waves. This method may not work for neutrinos, since neutrinos propagate with almost the light speed due to their small masses and are detected at a distant position. Hence the neutrino wave produced in a pion decay is similar to an electrostatic potential of a moving body, which has a finite-size correction in a form of a retarded potential.

A wave function that starts from an eigenstate of a free Hamiltonian H_0 of an eigenvalue E_0 and evolves with a total Hamiltonian $H_0 + H_1$, where H_1 is an interaction Hamiltonian, is a superposition of waves of various kinetic energies E_β at a finite time. So the wave function expressing the whole process approaches an asymptotic form slowly in certain situations. For computing a probability in this region, the ordinary method of S-matrix, which we write $S[\infty]$, is useless and those that satisfy the boundary condition at a finite-time interval T and wave packets localized in space that fulfill boundary conditions of experiments are necessary. An S-matrix $S[T]$ that satisfies the boundary conditions at a finite T is appropriate to study them. We develop a theory for $S[T]$ and compute the finite-size corrections of transition probabilities with $S[T]$ expressed by wave packets.

An amplitude to detect a particle at a finite distance or time is written with suitable wave packets for the final states in LSZ [14] reduction formula. When calculations are made without replacing the wave packets with plane waves, an expression is obtained and the probability is computed. To study the probability that depends on the position is unfamiliar in high-energy physics, hence we review an essence of the method first. Let $\psi(x, \gamma; \vec{X}^{(i)})$ be a wave function at a space-time position x of a particle that we observe and other particles γ in a scattering process, where $\vec{X}^{(i)}$ shows the position of the target. $\psi(x, \gamma; \vec{X}^{(i)})$ is expressed by a corresponding Green's function and its explicit form for a detection of a pion will be given later and that for a neutrino will be given in II. The amplitude to detect the particle of a momentum \vec{p} at X is written as

$$\Psi(X, \vec{p}, \gamma) = \int d^4x w(\vec{p}, x - X) \psi(x, \gamma; \vec{X}^{(i)}), \quad (3)$$

where $w(\vec{p}, x - X)$ is a wave packet that satisfies a wave equation of the particle and $\vec{X}^{(i)}$ is omitted in $\Psi(X, \vec{p}, \gamma)$. A Gaussian form with a size σ of a momentum \vec{p} at a position X^μ

$$w(\vec{p}, x - X) = N_0 e^{-(\vec{x} - \vec{X} - \vec{v}(t - X^0))^2 / \sigma - i\vec{p} \cdot (x - X)}, \quad (4)$$

is used for the sake of simplicity, where N_0 is the normalization factor and \vec{v} is a velocity of the relativistic particle $v_i = \frac{\partial}{\partial p_i} E(\vec{p}) = p_i / E(\vec{p})$. We write a four-dimensional coordinate as x and its components as (x^0, \vec{x}) . For the 0-th component x^0 , sometimes t is used, $x^0 = t$.

An integral over the states γ of the square of modulus of this amplitude

$$C(X, \vec{p}) = \int d\gamma \Psi^*(X, \vec{p}; \gamma) \Psi(X, \vec{p}; \gamma), \quad (5)$$

is a probability of detecting this particle at X , where four dimensional coordinates x are integrated over $X^{\mu(i)} \leq x^\mu$. $X^{(i)}$ is the coordinate of the target particle, and $d\gamma$ expresses the whole phase space of the state γ .

$C(X, \vec{p})$ is written with a correlation function $\Delta(x_1, x_2)$ defined from the wave function $\psi(x, \gamma; \vec{X}^{(i)})$ as

$$\Delta(x_1, x_2) = \int d\gamma \psi^*(x_1, \gamma; \vec{X}^{(i)}) \psi(x_2, \gamma; \vec{X}^{(i)}). \quad (6)$$

This function, in a relativistic invariant system, has a term of the form,

$$\int \frac{d\vec{p}}{E(\vec{p})} e^{i(p+Q) \cdot (x_1 - x_2)}, \quad (7)$$

where p_μ is the four-dimensional momentum of the mass shell of a particle and Q_μ is a momentum of others. This integral is known to be composed of a term proportional to the singular function [17]

$$\delta(\lambda) \epsilon(t_1 - t_2), \lambda = (x_1 - x_2)^2, \quad (8)$$

$$\epsilon(t_1 - t_2) = \pm 1 \text{ for } t_1 - t_2 \gtrless 0, \quad (9)$$

and those proportional to the regular functions $e^{i\tilde{m}\lambda}$, $e^{-\tilde{m}\lambda}$ where a finite \tilde{m} is determined from a dynamics of the system.

$C(X, \vec{p})$ is expressed with $\Delta(x_1, x_2)$ as

$$C(X, \vec{p}) = \int \prod_{i=1,2} d^4 x_i w^*(\vec{p}, x_1 - X) w(\vec{p}, x_2 - X) \Delta(x_1, x_2), \quad (10)$$

and behaves as

$$C^{(0)} + C^{(1)}\tilde{g}(|\vec{X} - \vec{X}^{(i)}|/l_0), \quad (11)$$

where $C^{(0)}$ and $C^{(1)}$ depend on the momenta and a function $\tilde{g}(|\vec{X} - \vec{X}^{(i)}|/l_0)$ is studied in Section 3 and decreases rapidly in $\frac{|\vec{X} - \vec{X}^{(i)}|}{l_0} \geq 10$, and behaves as $\frac{2l_0}{|\vec{X} - \vec{X}^{(i)}|}$ at $\frac{|\vec{X} - \vec{X}^{(i)}|}{l_0} \rightarrow \infty$. The rapidly oscillating or decreasing terms in $\Delta(x_1, x_2)$ contribute to the constant term, $C^{(0)}$, and the singular term, Eq.(8), gives the second term of the above equation expressing a finite-size correction. l_0 determines a typical length that the finite-size correction remains and is called a coherence length of the wave function. Thus the singular function gives the long-range correlation.

From explicit calculations which are presented later, a form of l_0 in a high momentum region, $|\vec{p}| \gg m$, is given by

$$l_0 = \left(\frac{2|\vec{p}|\hbar c}{m^2} \right), \quad (12)$$

where m is the particle's mass. For a pion and electron of energy 1 [GeV], they are

$$l_0^{pion} = \frac{2\hbar c}{0.13^2}[\text{GeV}^{-1}] = 2 \times 10^{-14}[\text{m}], \quad (13)$$

$$l_0^{electron} = \frac{2\hbar c}{0.5^2}[\text{GeV}^{-1}] = 1.3 \times 10^{-9}[\text{m}]. \quad (14)$$

Other hadrons are heavier and have shorter lengths than that of the pion. Thus l_0 for hadrons and charged leptons are microscopic lengths.

In the asymptotic regions where initial and final states are expressed by free particles, $C(X, \vec{p})$ neither depends on the longitudinal coordinate $(\vec{X} - \vec{X}^{(i)})_L$ parallel to the momentum \vec{p} nor the time $X^0 - X^{0(i)}$ and the wave behaves like a particle and the standard S-matrix is applied in this space-time region. Processes of detecting particles in $|\vec{X} - \vec{X}^{(i)}| \gg l_0$ are in the asymptotic region. Since the particle in this region is expressed by a plane wave and behaves like a free particle, the particle number and flux behave like those of classical particles and the production and detection are treated separately. The particle flux is determined by its distribution functions of the number and velocity that are determined by the decay process and is used for calculation of the scattering processes. From Eq. (14), scattering experiments

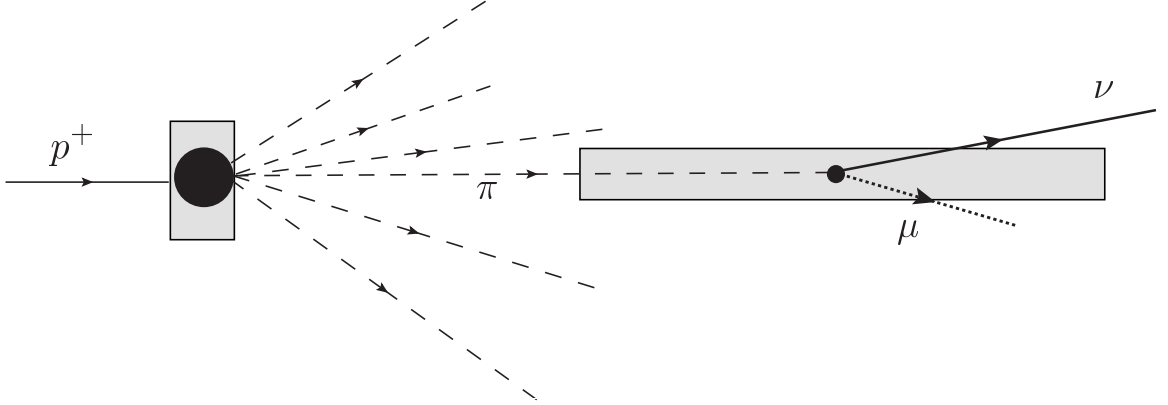


Fig. 1: The whole process in a high-energy neutrino experiment is illustrated. By a collision of a proton with a target, a pion is produced. The pion propagates a macroscopic distance and decays in a decay tunnel. A neutrino is produced and is detected.

of the charged leptons and hadrons using detectors of macroscopic sizes and distances are treated with the ordinary S-matrix. The transition amplitude has a delta function of an energy and momentum conservation and a total probability becomes proportional to a large time interval T and a decay rate and a mean life time are found from the probability per time, i.e., a proportional constant.

If $C(X, \vec{p})$ varies with $(\vec{X} - \vec{X}^{(i)})_L$ or $X^0 - X^{0(i)}$, the final state behaves like a correlated wave and is not in the asymptotic region. The process then is not described by the standard S-matrix of plane waves but by a Schrödinger equation of a suitable boundary condition, which describes a time-evolution of the state vector regardless of the boundary conditions, or by the S-matrix expressed by wave packets and satisfies a boundary condition at a finite T , $S[T]$.

The whole process of the production and decay of a pion is expressed in Fig. 1. The first reaction is caused by strong interactions and the second reaction is caused by weak interactions. These interactions are understood well and scattering probabilities, cross sections, and other physical quantities which are defined at asymptotic regions and depend on particles' momenta of the initial and final states have been computed theoretically and agreements with experiments have been obtained.

Here we study new possibilities that the particles are detected in non-

asymptotic regions where $C(X, \vec{p})$ varies with $(\vec{X} - \vec{X}^{(i)})_L$ or $X^0 - X^{0(i)}$. The pion is studied in I and the neutrino is studied in II. We find that the pion retains wave nature in short distance region and is detected in the asymptotic region but the neutrino retains wave nature in a wide area and is detected in the non-asymptotic space-time regions in certain area. A probability to detect a neutrino varies with the distance and has a large finite-size correction, in fact. Being described by a many-body wave function together with the charged lepton and pion, the neutrino is not a simple isolated wave but is a superposed wave of many components at a position of detector. The neutrino is detected through its interaction with a nucleus in a target. Hence the probability to detect neutrinos at a finite distance is computed with wave packets that are localized in space. The wave packets vanish at a position \vec{x} if the distance between \vec{x} and the center position \vec{X} , $|\vec{x} - \vec{X}| \rightarrow \infty$, and satisfy the boundary conditions of the experiments. Hence they are appropriate to study the finite-size effect and are used here [14, 15].

This paper is organized in the following manner. In section 2, S-matrix of a finite-time interval, $S[T]$, is introduced. In section 3, pions in hadron reactions are studied. Summary and prospects are given in section 4. Various properties of wave packets including the size, shape, and completeness are studied in Appendix. The neutrino in the pion decay is studied in II.

2 Finite-size correction and S-matrix at a finite-time interval

In a scattering amplitude, an initial state is prepared at $t = -\infty$ and a final state is observed at $t = +\infty$. In practical situations, these times are not infinite but large values $\pm T/2$. Approximation of the finite T with ∞ is normally good and a deviation of physical quantity between those of T and of ∞ , which we call a finite-size correction, is negligibly small then. That is not the case in some situations. So, general properties of wave function and scattering matrix at the finite-time interval are studied, first.

2.1 Wave function at a finite time

2.1.1 Wave functions at a finite time

In a physical system described by a Hamiltonian H composed of a free term H_0 and an interaction term H_1

$$H = H_0 + H_1, \quad (15)$$

the Schrödinger equation in a unit of $\hbar = 1$,

$$i \frac{\partial}{\partial t} |\Psi(t)\rangle = (H_0 + H_1) |\Psi(t)\rangle, \quad (16)$$

describes the time evolution. Unitary operators

$$U(t) = e^{-iHt}, \quad U_0(t) = e^{-iH_0t}, \quad (17)$$

give time evolutions of the state vectors. State vector in the interaction representation defined by

$$|\tilde{\Psi}(t)\rangle = U_0 |\Psi(t)\rangle, \quad \tilde{H}_1(t) = U_0(t) H_1 U_0^\dagger(t), \quad (18)$$

satisfies

$$i \frac{\partial}{\partial t} |\tilde{\Psi}(t)\rangle = \tilde{H}_1(t) |\tilde{\Psi}(t)\rangle, \quad (19)$$

and a solution is given by a time-ordered product,

$$|\tilde{\Psi}(t)\rangle = T \int_0^t dt' e^{\tilde{H}_1(t')/i} |\Psi(0)\rangle = |\Psi(0)\rangle + \int_0^t dt' \tilde{A}(t') |\Psi(0)\rangle, \quad (20)$$

where

$$\tilde{A}(t') = \tilde{H}_1(t')/i + \int_0^{t'} dt'' (\tilde{H}_1(t')/i) (\tilde{H}_1(t'')/i + \dots). \quad (21)$$

In the first order in H_1 , there is no ambiguity of higher order corrections and the solution is

$$|\tilde{\Psi}(t)\rangle = \left\{ 1 + \int_0^t dt' \tilde{H}_1(t')/i \right\} |\Psi(0)\rangle. \quad (22)$$

and is written as

$$\begin{aligned}
&= |\Psi(0)\rangle + \int_0^t dt' \int d\beta |\beta\rangle \langle \beta | \tilde{H}_1(t') / i | \Psi(0)\rangle \\
&= |\Psi(0)\rangle + \int d\beta \frac{e^{i\omega t} - 1}{\omega} |\beta\rangle \langle \beta | \tilde{H}_1(0) | \Psi(0)\rangle,
\end{aligned} \tag{23}$$

where

$$H_0 |\Psi(0)\rangle = E_0 |\Psi(0)\rangle, \quad H_0 |\beta\rangle = E_\beta |\beta\rangle, \tag{24}$$

and

$$\omega = E_\beta - E_0. \tag{25}$$

At $t \rightarrow \infty$, the formula

$$\frac{e^{i\omega t} - 1}{\omega} = 2ie^{i\frac{\omega t}{2}} \left(\frac{\sin(\omega t/2)}{\omega} \right) \approx 2\pi i \delta(\omega), \tag{26}$$

is substituted, and the states of $E_\beta = E_0$ contribute. But at a finite, t states of broad spectrum contribute. At a finite ω , the exponential function oscillates rapidly and its average in δt of satisfying $\omega \delta t \gg 1$ is given by

$$\frac{e^{i\omega t} - 1}{\omega} \approx -\frac{1}{\omega}. \tag{27}$$

The wave function Eq. (23) at the infinite t and the average at a finite t agree with

$$|\tilde{\Psi}(t)_\infty\rangle = |\Psi(0)\rangle + 2\pi i \int d\beta |\beta\rangle \langle \beta | \tilde{H}_1(0) | \Psi(0)\rangle \delta(E_\beta - E_0), \tag{28}$$

$$|\tilde{\Psi}(t)_{average}\rangle = |\Psi(0)\rangle - \int d\beta \frac{1}{\omega} |\beta\rangle \langle \beta | \tilde{H}_1(0) | \Psi(0)\rangle; \text{finite } t. \tag{29}$$

Both of them have the total energy E_0 of the total Hamiltonian H

$$H |\tilde{\Psi}(t)\rangle = E_0 |\Psi(t)\rangle. \tag{30}$$

They have different energies of H_0

$$H_0 |\tilde{\Psi}(t)_\infty\rangle = E_0 |\Psi(t)_\infty\rangle, \tag{31}$$

$$H_0 |\tilde{\Psi}(t)_{average}\rangle \neq E_0 |\Psi(t)_{average}\rangle. \tag{32}$$

It is noted that the energy E_β of H_0 in Eq. (28) is E_0 and the kinetic energy defined by the free part H_0 is conserved at the infinite time, whereas E_β in Eq. (23) is in a wide range of certain weight and a mean value of E_β varies with time and differs from E_0 . The difference is due to H_1 . Hence the energy defined by the free part H_0 is not conserved at a finite time but the difference $E_\beta - E_0$ vanishes in the asymptotic region.

Thus states at $t \rightarrow \infty$ are governed by the states of the same kinetic energy E_0 and the asymptotic values such as the total cross section and decay rate are computed within this space. Now the state at a finite t , Eq. (23), is completely different and is composed of many components of eigenvalues $E_\beta \neq E_0$. This state is a superposition of waves of different kinetic energies with a definite weight and is similar to electron waves formed by interferometer such as that in a bi-prism experiment. Plane waves of various kinetic energies are generated by potential energies of an obstacle or potential in a stationary wave and show a clear single-electron interference phenomenon.

Here waves of various kinetic energies are generated by the interaction energy due to H_1 in a time-dependent wave function. This many-body interaction initially makes a transition of a particle to two particles if that is a form of ϕ^3 or from two particle to two particle if that is a form of ϕ^4 and gives an energy to a state at a finite t of Eq. (23) as a by-product. Since the total energy is conserved, the kinetic energy is not constant. Accordingly, the wave Eq. (23) could show a diffraction phenomenon that is characteristic of the waves of many components of certain weight. This diffraction pattern depends on t and gives a finite-size correction to the probability. They depend upon the spectrum and states at all E_β , even this is the phenomenon of the wave function at the tree level.

The weight of $|\beta\rangle$ in wave functions at a finite t and $t = \infty$ are proportional to $\frac{e^{i\omega t} - 1}{\omega}$ and $2\pi i\delta(\omega)$ in any order of H_1 . Thus the waves are superpositions of waves of all E_β at a finite t , and superpositions of waves of $E_\beta = E_0$ at $t = \infty$.

2.1.2 Scattering operator of a finite-time interval

The wave function is not directly observable. Observables are measured with scattering or decay processes of the particles and their amplitudes are described by Møller operators. Those at a finite-time interval T are defined

from $U(t)$ and $U_0(t)$ in the form

$$\Omega_{\pm}(T) = \lim_{t \rightarrow \mp T/2} U^{\dagger}(t) U_0(t), \quad (33)$$

and satisfy

$$e^{iH\tau} \Omega_{\mp}(T) = \Omega_{\mp}(T \pm \tau) e^{iH_0\tau}, \quad (34)$$

Taking an infinitesimal τ , we have

$$(1 + iH\tau) \Omega_{\mp}(T) = \left\{ \Omega_{\mp}(T) \pm \tau \frac{\partial}{\partial T} \Omega_{\mp}(T) \right\} (1 + iH_0\tau). \quad (35)$$

Thus the relation

$$H \Omega_{\mp}(T) = \mp i \frac{\partial}{\partial T} \Omega_{\mp}(T) + \Omega_{\mp}(T) H_0, \quad (36)$$

is satisfied. The first term in the right hand side is a specific term of those operators of the finite T . Hermitian conjugate of this equality is

$$\Omega_{\mp}^{\dagger}(T) H = \pm i \left(\frac{\partial}{\partial T} \Omega_{\mp}(T) \right)^{\dagger} + H_0 \Omega_{\mp}^{\dagger}(T). \quad (37)$$

Scattering operator of a finite-time interval T is the product

$$S(T) = \Omega_{-}^{\dagger}(T) \Omega_{+}(T), \quad (38)$$

and satisfies

$$\begin{aligned} S(T) H_0 &= \Omega_{-}^{\dagger}(T) \Omega_{+}(T) H_0 \\ &= \Omega_{-}^{\dagger}(T) \left\{ H \Omega_{+}(T) - i \frac{\partial}{\partial T} \Omega_{+}(T) \right\} \\ &= H_0 \Omega_{-}^{\dagger}(T) + i \left(\frac{\partial}{\partial T} \Omega_{-}(T) \right)^{\dagger} \Omega_{+}(T) - i \Omega_{-}^{\dagger}(T) \frac{\partial}{\partial T} \Omega_{+}(T) \\ &= H_0 S(T) + i \left(\frac{\partial}{\partial T} \Omega_{-}(T) \right)^{\dagger} \Omega_{+}(T) - i \Omega_{-}^{\dagger}(T) \frac{\partial}{\partial T} \Omega_{+}(T), \end{aligned} \quad (39)$$

from Eq. (36). Finally we have

$$[S(T), H_0] = i \left(\frac{\partial}{\partial T} \Omega_{-}(T) \right)^{\dagger} \Omega_{+}(T) - i \Omega_{-}^{\dagger}(T) \frac{\partial}{\partial T} \Omega_{+}(T). \quad (40)$$

$S(T)$ does not commute with H_0 and does not conserve the energy defined by H_0 . The matrix element has the conserving and non-conserving terms,

$$\langle \beta | S(T) | \alpha \rangle = \delta_{\beta, \alpha} + \delta_\epsilon (E_\alpha - E_\beta) f_{\alpha, \beta} + \delta f(T) \quad (41)$$

of the kinetic energy. Since the energy E_β of the second term of the right-hand side of Eq. (41) is different from that of the third term, the total transition probability becomes a sum of each probability. The second term gives constant physical quantities that are computable by the ordinary S-matrix of plane waves and the third term gives T-dependent corrections that are not computable by the ordinary S-matrix. A magnitude of δf depends on a dynamics of the system and a probability gets modified by this term at the finite T. When E_α and E_β are approximate energies of the states $|\alpha\rangle$ and $|\beta\rangle$, we have

$$(E_\alpha - E_\beta) \langle \beta | S(T) | \alpha \rangle = \langle \beta | O(T) | \alpha \rangle, \\ O(T) = i \left(\frac{\partial}{\partial T} \Omega_-(T) \right)^\dagger \Omega_+(T) - i \Omega_-^\dagger \frac{\partial}{\partial T} \Omega_+(T). \quad (42)$$

Hence

$$\delta f = \frac{1}{E_\alpha - E_\beta} \langle \beta | O(T) | \alpha \rangle, \quad (43)$$

and a probability from these states that have the energies E_β different from E_α is given in the form,

$$\sum_\beta |\delta f|^2 = \sum_\beta \left(\frac{1}{E_\alpha - E_\beta} \right)^2 |\langle \beta | O(T) | \alpha \rangle|^2 \geq 0, \quad (44)$$

where the equality is satisfied at $T \rightarrow \infty$. Thus the finite-size correction is derived from those states that have the energy $E_\beta \neq E_0$. The examples of the finite-size correction in the lowest order of H_1 that reflects the states at ultra-violet region will be presented later in I and in II.

The fact that the energy defined by a free part H_0 is not conserved in a transition process of a time interval T, even though the total energy defined by a total Hamiltonian H is conserved and the state β of the energy E_β satisfying $E_\beta \neq E_\alpha$ contributes is known well in quantum mechanics [18]. A probability of detecting a state β at T is given in the form,

$$|F_{\alpha, \beta}|^2 \frac{4 \sin^2 [(E_\beta - E_\alpha)T]}{(E_\beta - E_\alpha)^2}, \quad (45)$$

where $F_{\alpha,\beta}$ is the matrix element. A variance of the energy difference $\Delta E = E_\alpha - E_\beta$ satisfies an uncertainty relation

$$T\Delta E \geq \hbar. \quad (46)$$

In a system of continuous spectrum also, a formula

$$\lim_{T \rightarrow \infty} \frac{4 \sin^2 [(E_\beta - E_\alpha)T]}{(E_\beta - E_\alpha)^2} = \lim_{T \rightarrow \infty} 2\pi T \delta(E_\beta - E_\alpha), \quad (47)$$

is used normally. If a system has states of $|E_\beta - E_\alpha| = \epsilon$, then the uncertainty relation leads the time T must satisfy

$$T \geq \frac{\hbar}{\epsilon}. \quad (48)$$

Conversely at a time below this value, the amplitude and probability may be different from the asymptotic values, and may have large finite-size corrections. The large T behavior of Eq. (45) should be examined carefully and not be replaced with the asymptotic expression Eq. (47).

The formulas for a large T , Eqs. (26) and (47), used usually for computing the rate of the transition [19, 20] can not be applied to find the finite-size correction, because Taylor expansion of $g(\omega)$ in

$$\int d\omega g(\omega) \left(\frac{2 \sin[\omega T/2]}{\omega} \right)^2 = 2\pi T g(0) + O(1/T), \quad (49)$$

leads diverging integrals over ω to the non-leading terms. Hence we estimate the integral over the time directly.

Each state in initial or final states of $S[T]$ is defined by its position in addition to the momentum in field theory. The invariance under the translation is broken slightly, and $S[T]$ does not commute with the momentum operator

$$[S[T], \vec{P}] \neq 0. \quad (50)$$

From Eqs. (40) and (50), both of the energy and momentum are not conserved in $S[T]$. Finite-size corrections of the transition rate and others are computed with $S[T]$ of satisfying the boundary conditions and with the T -dependent expression Eq. (45). We will see that particle states at ultraviolet energy regions in a relativistic invariant system, which couple with the $S[T]$ in a universal manner, give the finite corrections.

2.2 Internal symmetry

When a conserved charge Q commutes with the total Hamiltonian

$$[Q, H] = 0, \quad (51)$$

and with a free part, H_0 ,

$$[Q, H_0] = 0, \quad (52)$$

then Q commutes with the S-matrix at the finite-time interval, $S(T)$,

$$[Q, S(T)] = 0. \quad (53)$$

Hence the charge connected with an internal symmetry is conserved in $S(T)$. Thus a state $|\psi\rangle$ and the transformed state $S[T]|\psi\rangle$ have a same charge

$$Q|\psi\rangle = q|\psi\rangle, \quad (54)$$

$$QS[T]|\psi\rangle = qS[T]|\psi\rangle. \quad (55)$$

Hence the charge is conserved in the transition process of the finite-time interval also.

If Q is a charge of non-compact group, its eigenvalue

$$Q|q\rangle = q|q\rangle, \quad (56)$$

is continuous and the eigenstates are normalized with Dirac delta function,

$$\langle q_1 | q_2 \rangle = 2\pi\delta(q_1 - q_2). \quad (57)$$

Then the matrix element is written in the diagonal form in q ,

$$\langle q_2 | S[T] | q_1 \rangle = 2\pi\delta(q_1 - q_2)\tilde{S}[T](q_1), \quad (58)$$

and the matrix element between any states is written with the reduced matrix $\tilde{S}[T]$

$$\int dq_2 dq_1 F(q_2) \langle q_2 | S[T] | q_1 \rangle G(q_1) = 2\pi \int dq_1 F(q_1) \tilde{S}[T](q_1) G(q_1). \quad (59)$$

2.3 Unitarity

The $S[T]$ is defined with Møller operators, Eq. (38), and satisfies a unitarity relation,

$$S^\dagger[T]S[T] = S[T]S^\dagger[T] = 1, \quad (60)$$

and an optical theorem

$$i(T[T] - T[T]^\dagger) = T[T]T[T]^\dagger, \quad (61)$$

$$S[T] = 1 + iT[T]. \quad (62)$$

The probability is preserved in $S[T]$ and the imaginary part of the amplitude at T is determined by the total probability measured at T.

$S[T]$ is decomposed into the energy conserving term $T_1[T]$ and non-conserving term $T_2[T]$

$$S[T] = 1 + i(T_1[T] + T_2[T]), \quad (63)$$

then the unitarity Eq. (60) is written in the form,

$$(1 + i(T_1[T] + T_2[T]))(1 - i(T_1^\dagger[T] + T_2^\dagger[T])) = 1. \quad (64)$$

Hence we have

$$\begin{aligned} -i(T_1[T] - T_1^\dagger[T]) - i(T_2[T] - T_2^\dagger[T]) &= T_1[T]T_1^\dagger[T] + T_2[T]T_2^\dagger[T] \\ &\quad + T_1[T]T_2^\dagger[T] + T_2[T]T_1^\dagger[T], \end{aligned} \quad (65)$$

and

$$-i(T_1[T] - T_1^\dagger[T]) = T_1[T]T_1^\dagger[T], \quad (66)$$

$$-i(T_2[T] - T_2^\dagger[T]) = T_2[T]T_2^\dagger[T] + T_1[T]T_2^\dagger[T] + T_2[T]T_1^\dagger[T]. \quad (67)$$

The total transition probability from a state α is

$$P = \sum_{E_\beta \approx E_\alpha} |\langle \beta | T_1 | \alpha \rangle|^2 + \sum_{E_\beta \neq E_\alpha} |\langle \beta | T_2 | \alpha \rangle|^2, \quad (68)$$

where the energy conserving term is expressed by

$$\langle \beta | T_1 | \alpha \rangle = \langle \beta | \tilde{T}_1 | \alpha \rangle 2\pi\delta(E_\beta - E_\alpha). \quad (69)$$

It is noted that the unitarity connects physical quantities measured at T. Optical theorem proves that the imaginary part of forward amplitudes at T is written by total probability at T. Hence the life time at T, depends on T if the finite-size correction to the total probability is finite. The unitarity does not connect the probability at T with those at a different T.

2.4 Fluctuations

Fluctuations of elementary particles are specified by Compton wave length $L_{cw} = \frac{ch}{m}$, where m is a particle's mass. L_{cw} for various particles are

$$L_{cw} \geq \begin{cases} 2 \times 10^{-7} \text{ [m]} & \text{neutrino } (m_\nu \leq 1\text{eV}/c^2); \\ 4 \times 10^{-13} \text{ [m]} & \text{electron}; \\ 1 \times 10^{-15} \text{ [m]} & \text{muon}; \\ 2 \times 10^{-15} \text{ [m]} & \text{pion}; \\ 2 \times 10^{-12} \text{ [m]} & \text{proton}. \end{cases} \quad (70)$$

They are short range.

In a relativistic field theory, a fluctuation of a field is expressed with a Green's function. A wave produced at a space-time position x_1 and observed at another position x_2 is a Green's function, $\Delta(x_1 - x_2)$ and is a function of $x_1 - x_2$ owing to a translational invariance. $\Delta(x_1 - x_2)$ is written in the Fourier transformation

$$\Delta(x_1 - x_2) = \frac{1}{(2\pi)^4} \int d^4p e^{ip \cdot (x_1 - x_2)} \frac{1}{p^2 - m^2}, \quad (71)$$

where an integration in p^0 is taken along a complex path which avoids a pole of the integrand. From the residue of the pole, we have a component of on-mass shell waves of positive energy

$$\Delta_0(x_1 - x_2) = \frac{1}{(2\pi)^3} \int \frac{d\vec{p}}{2E(\vec{p})} e^{ip \cdot (x_1 - x_2)}, \quad E(\vec{p}) = \sqrt{\vec{p}^2 + m^2}. \quad (72)$$

$\Delta_0(x_1 - x_2)$ is composed of (1) a singular part that is proportional to the Dirac delta function of the form $\delta(\lambda)$, $\lambda = (x_1 - x_2)^2$ and (2) Bessel functions. The former one of (1) is narrow in λ and long-range in $|t_1 - t_2|$ or $|\vec{x}_1 - \vec{x}_2|$ and is called light-cone singularity. The light-cone singularity is characteristic of relativistic invariant systems. Since a one-particle energy of a mass m and a momentum \vec{p} is $E(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$ and approaches $E(\vec{p}) \rightarrow |\vec{p}|$ in a large momentum region, the phase in Eq. (72), $p \cdot (x_1 - x_2)$, cancels at a light-cone, $|t_1 - t_2| = |\vec{x}_1 - \vec{x}_2|$ in the direction \vec{p} in the infinite momentum. The wave becomes singular and real along the light cone and gives important contributions to a probability at a finite distance. The latter ones (2) are short-range functions which are specified by a length similar to Eq. (70) and describe short-range fluctuations.

2.5 Boundary conditions

Boundary conditions are necessary to determine a solution of a wave equation uniquely but is not explicitly written sometimes. In scattering processes, asymptotic boundary conditions are important [14]. For a scattering of a scalar field from an initial state $|\alpha\rangle$ to a final state $|\beta\rangle$, the states $|\alpha\rangle$ at $t = -\infty$ are constructed with free waves and the states $|\beta\rangle$ at $t = \infty$ are constructed also with free waves and satisfy asymptotic boundary conditions,

$$\lim_{t \rightarrow -\infty} \langle \alpha | \phi^f(t) | \beta \rangle = \langle \alpha | \phi_{in}^f | \beta \rangle, \quad (73)$$

$$\lim_{t \rightarrow \infty} \langle \alpha | \phi^f(t) | \beta \rangle = \langle \alpha | \phi_{out}^f | \beta \rangle. \quad (74)$$

In the above equations, the field operators, $\phi^f(x)$, $\phi_{in}^f(x)$, and $\phi_{out}^f(x)$ are expanded with a complete set of normalized functions $f(\vec{x}, t)$, and the coefficient $\phi^f(x)$ is given in the form,

$$\phi^f(t) = i \int d^3x f^*(\vec{x}, t) \partial_0 \phi(\vec{x}, t), \quad (75)$$

and the same coefficients $\phi_{in}^f(x)$ and $\phi_{out}^f(x)$ are defined in the equivalent manner.

To study a scattering at a finite-time interval T , asymptotic conditions are modified to

$$\lim_{t \rightarrow -T/2} \langle \alpha | \phi^f(t) | \beta \rangle = \langle \alpha | \phi_{in}^f | \beta \rangle, \quad (76)$$

$$\lim_{t \rightarrow +T/2} \langle \alpha | \phi^f(t) | \beta \rangle = \langle \alpha | \phi_{out}^f | \beta \rangle. \quad (77)$$

In the above equations, the fields $\phi_{in}^f(x)$ and $\phi_{out}^f(x)$ and the function $f(\vec{x}, t)$ satisfy the free wave equation and the states $|\alpha\rangle$ and $|\beta\rangle$ are defined with wave packets. Since the wave packets have finite spatial sizes and decrease fast at large $|\vec{x} - \vec{x}_0|$ around a center \vec{x}_0 , they ensure the asymptotic conditions at a finite T . It is important to satisfy the conditions even at a finite T to compute the finite-size corrections that are compared with experiments. We will see that in fact the finite-time interval corrections are obtained with wave packets.

3 Coherence length and asymptotic behavior of pion

A charged pion is produced in a hadron reaction and decays to a neutrino and a charged lepton, after it propagates a certain distance. A pion is described with a wave field and its wave nature in the reactions is determined by the wave equation and the transition amplitude. We study hadron reactions using a Lagrangian of the pion and nucleon fields and analyze a position-dependent probability of detecting a pion at a finite distance. It is found that the position dependence of the probability is determined by the pion's mass and energy.

Medium effects and detector effects are studied in Appendix and the pion's decay is studied in II.

3.1 Pion emitted from a nucleon

Pions and nucleons are an iso-triplet and an iso-doublet, and are expressed with fields $\vec{\phi}(x)$ and $N(x)$. This physical system is described in term of the renormalizable Lagrangian,

$$\mathcal{L} = \bar{N}(\gamma \cdot p - m_N)N + g\bar{N}\gamma_5\vec{\tau} \cdot \vec{\phi}(x)N + \frac{1}{2}(\partial_\mu\vec{\phi})^2 - \frac{1}{2}m_\pi^2\vec{\phi}^2(x), \quad (78)$$

where m_N and m_π are masses of the nucleons and pions. A mass difference between a proton and a neutron and that of neutral and charged pions are ignored and $SU(2)$ symmetry is assumed in most places. Second term in the right-hand side shows an interaction between a nucleon and a pion. Due to this interaction, a nucleon emits and absorbs a pion in intermediate states. These physical processes are treated by a renormalization prescription in a nucleon self-energy and others. The self-energy has no imaginary part at a mass-shell momentum since one nucleon does not decay and is stable. These particles in asymptotic states, which are defined at $t = \pm\infty$, do not decay due to the energy-momentum conservation. A nucleon at $t = -\infty$ does not transform into a nucleon and a pion at $t = +\infty$. Nevertheless, one pion and one nucleon can exist for a short period as a virtual state. So this pion could be detected with a finite probability in a small space-time area using a small apparatus. Hereafter a space-time area where one pion is observed with a finite probability is estimated.

Let a nucleon of a momentum \vec{p}_{N_i} set at time $t = t_i$ make a transition to a pion of average values of a three dimensional momentum \vec{p}_π at a four dimensional position (T_π, \vec{X}_π) and other particles. A pion wave function of having these properties is $w(\vec{p}, X)$ of Eq. (4). This set of functions is complete set [21] and is equivalent to that of plane waves mathematically. Now the amplitude expressed with $w(\vec{p}, X)$ satisfies the boundary condition at T, and is a part of $S[T]$, hence is different from the amplitude expressed with the plane waves that satisfies the boundary condition at $T = \infty$. The probability of measuring the momentum and position with a small detector composed of nucleus of the transition process is computed with this amplitude expressed with wave packets. The wave packet size σ_π is a size of wave function that the pion interacts with.

In a transition amplitude of a nucleon of the momentum \vec{p}_{N_i} at $t = T_{N_i}$ to a nucleon of the momentum \vec{p}_{N_f} and a pion,

$$T = \int d^4x \langle N_f, pion | H_i(x) | N_i \rangle, H_i = g \bar{N} \gamma_5 \vec{\tau} \cdot \vec{\phi}(x) N, \quad (79)$$

the time and space are integrated over the region $T_{N_i} \leq x^0 \leq T_\pi, X_{N_i}^j \leq x^j \leq X_\pi^j$. The time interval is finite and this amplitude receives contributions from both of conserving and non-conserving states of the kinetic energy, from Eqs. (23) and (40). The former is a normal term and agrees with that computed with S-matrix of plane waves and the latter is a correction term that vanishes at $T \rightarrow \infty$ and is not computable with S-matrix of plane waves but with $S[T]$.

The initial and final states are described by plane waves or the wave packet of the central values of the momentum and coordinate, and the width in the form

$$|N_i\rangle = |\vec{p}_{N_i}, T_{N_i}\rangle, |N_f, pion\rangle = |N_f, \vec{p}_{N_f}; pion, \vec{p}_{pion}, \vec{X}_{pion}, T_{pion}\rangle. \quad (80)$$

The matrix elements of these states are defined in the forms,

$$\begin{aligned} \langle \vec{p}_\pi, \vec{X}_\pi, T_\pi | \varphi(x) | 0 \rangle &= N_\pi \rho_\pi \int d\vec{k}_\pi e^{-\frac{\sigma_\pi}{2}(\vec{k}_\pi - \vec{p}_\pi)^2} e^{i(E(\vec{k}_\pi)(t - T_\pi) - i\vec{k}_\pi \cdot (\vec{x} - \vec{X}_\pi))} \\ &\approx N_\pi \rho_\pi \left(\frac{2\pi}{\sigma_\pi} \right)^{\frac{3}{2}} e^{-\frac{1}{2\sigma_\pi}(\vec{x} - \vec{X}_\pi - \vec{v}_\pi(t - T_\pi))^2} e^{i(E(\vec{p}_\pi)(t - T_\pi) - \vec{p}_\pi \cdot (\vec{x} - \vec{X}_\pi))}, \end{aligned} \quad (81)$$

$$\begin{aligned} \langle N_f, \vec{p}_{N_f} | \bar{u}(x) \gamma_5 u(x) | N_i, \vec{p}_{N_i}, T_{N_i} \rangle &= \frac{1}{(2\pi)^3} \left(\frac{m_N}{E(\vec{p}_{N_f})} \right)^{\frac{1}{2}} \left(\frac{m_N}{E(\vec{p}_{N_i})} \right)^{\frac{1}{2}} \frac{1}{\sqrt{V_i}} \\ &\times \bar{u}(\vec{p}_{N_f}) \gamma_5 u(\vec{p}_{N_i}) e^{i((E(\vec{p}_{N_f}) - E(\vec{p}_{N_i}))t - (\vec{p}_{N_f} - \vec{p}_{N_i}) \cdot \vec{x} + E(\vec{p}_{N_i})T_{N_i})}, \end{aligned} \quad (82)$$

where

$$N_\pi = \left(\frac{\sigma_\pi}{\pi} \right)^{\frac{3}{4}}, \quad \rho_\pi = \left(\frac{1}{2E_\pi(2\pi)^3} \right)^{\frac{1}{2}}, \quad V_i : \text{volume for initial state.}$$

In this paper, the spinor's normalization is

$$\sum_s u(p, s) \bar{u}(p, s) = \frac{\gamma \cdot p + m}{2m}. \quad (83)$$

In the above equation the pion life time is ignored. σ_π in Eq. (81) is the size of the pion wave packet. Minimum wave packets are used in most of the present paper but non-minimum wave packets give the equivalent result, which are presented in II.

Substituting Eq. (81), we have the amplitude, Eq. (79), in the form,

$$\begin{aligned} N \int dt d\vec{x} d\vec{k}_\pi e^{-i(\vec{p}_{N_i} - \vec{p}_{N_f} - \vec{k}_\pi) \cdot \vec{x}} e^{i(E(\vec{p}_{N_f}) - E(\vec{p}_{N_i}) + E(\vec{k}_\pi))t + iE(\vec{p}_{N_i})T_{N_i}} \\ \times \bar{u}(\vec{p}_{N_f}) \gamma_5 u(\vec{p}_{N_i}) e^{-\frac{\sigma_\pi}{2}(\vec{k}_\pi - \vec{p}_\pi)^2} e^{-i(E(\vec{k}_\pi)T_\pi - \vec{k}_\pi \cdot \vec{X}_\pi)} \\ = N \int dt d\vec{k}_\pi (2\pi)^3 \delta_L(\vec{p}_{N_i} - \vec{p}_{N_f} - \vec{k}_\pi) e^{i(E(\vec{p}_{N_f}) - E(\vec{p}_{N_i}) + E(\vec{k}_\pi))t + iE(\vec{p}_{N_i})T_{N_i}} \\ \times \bar{u}(\vec{p}_{N_f}) \gamma_5 u(\vec{p}_{N_i}) e^{-\frac{\sigma_\pi}{2}(\vec{k}_\pi - \vec{p}_\pi)^2} e^{-i(E(\vec{k}_\pi)T_\pi - \vec{k}_\pi \cdot \vec{X}_\pi)}, \end{aligned} \quad (84)$$

$$N = N_\pi \rho_\pi \left(\frac{m_N}{(2\pi)^3 E(\vec{p}_{N_f})} \frac{m_N}{(2\pi)^3 E(\vec{p}_{N_i})} \frac{1}{\sqrt{V_i}} \right)^{\frac{1}{2}},$$

where $\delta_L(x)$ is an approximate delta function of a finite size L, where $L = |\vec{X}_\pi - \vec{X}_{N_i}|$. The real part of the exponent becomes stationary at the center of Gaussian function,

$$\vec{k}_\pi = \vec{p}_\pi. \quad (85)$$

Combing this momentum with

$$\vec{p}_{N_f} = \vec{p}_{N_i} - \vec{k}_\pi, \quad (86)$$

the \vec{p}_{N_f} satisfies

$$\vec{p}_{N_f} \approx \vec{p}_{N_i} - \vec{p}_\pi, \quad (87)$$

and the integration around this momentum has a finite contribution to the probability. Since momentum is conserved within uncertainty $\frac{1}{\sqrt{\sigma_\pi}}$, that is exactly conserved in the limit, $\sigma_\pi \rightarrow \infty$. So the probability from this kinematical region is regarded as an energy-conserving term that is computed in the standard method of plane waves and asymptotic boundary conditions. This probability is independent of σ_π and agrees with the value obtained by the ordinary S-matrix. Thus a small violation of the energy-momentum conservation does not modify the total probability of the normal component.

Now there is another stationary momentum that is obtained from the imaginary part of the exponent. At large $T_\pi - T_{N_i}$ and $|\vec{X}_\pi - \vec{X}_{N_i}|$, the imaginary part of the exponent becomes large and the amplitude oscillates rapidly unless it satisfies the stationary condition,

$$\frac{\partial}{\partial \vec{k}_\pi} \xi = 0, \quad \xi = ((E(\vec{p}_{N_f}) - E(\vec{p}_{N_i}) + E(\vec{k}))t - E(\vec{k})T_\pi + \vec{k} \cdot \vec{X}_\pi)|_{\vec{k}=\vec{p}_{N_i}-\vec{p}_{N_f}}. \quad (88)$$

The solution is,

$$\begin{aligned} \vec{v}_{N_f}t - \vec{v}_\pi t + \vec{v}_\pi T_\pi - \vec{X}_\pi &= 0, \\ \vec{v}_{N_f} &= \frac{\vec{p}_{N_f}}{E(\vec{p}_{N_f})}, \vec{v}_\pi = \frac{\vec{p}_{N_i} - \vec{p}_{N_f}}{E(\vec{p}_{N_i} - \vec{p}_{N_f})}. \end{aligned} \quad (89)$$

The second stationary momentum Eq.(89) is different from Eq.(86) and causes a new contribution that becomes important at larger $T_\pi - T_{N_i}$ and $|\vec{X}_\pi - \vec{X}_{N_i}|$. The second stationary momentum is determined with the space-time position and \vec{k}_π could deviate from \vec{p}_π . Hence the violation of the energy and momentum conservation becomes large. Thus two stationary momenta have different properties. First one does not depend upon the time but the second one does and they lead different phases. The violation of the energy and momentum conservation is small in the first one but is rather large in the second one, hence the first one corresponds to the normal term of Eq.(41)

and the second one corresponds to the finite-size correction δf of Eq. (41). The two terms of different phases lead the probability to have a large finite-size correction. We will see next that the finite-size correction is computed rigorously and has a universal property.

3.2 Expressing probability with correlation function and light-cone singularity

The total probability is expressed in the form

$$P = \int d\vec{X}_\pi \frac{1}{V_i} \frac{1}{E_{N_i}(2\pi)^3} \frac{d\vec{p}_\pi}{E_\pi(2\pi)^3} \tilde{P}, \quad (90)$$

where \tilde{P} is written from Eq. (79),

$$\tilde{P} = \int dx_1 dx_2 \Delta(x_1, x_2) e^{-\sum_i \frac{1}{2\sigma_\pi} (\vec{x}_i - \vec{X}_\pi - \vec{v}(t_i - T_\pi))^2} e^{iE_\pi(\vec{p}_\pi)(t_1 - t_2) - i\vec{p}_\pi \cdot (\vec{x}_1 - \vec{x}_2)}, \quad (91)$$

with a correlation function

$$\Delta(x_1, x_2) = \int N d\vec{p}_{N_f} d(\vec{p}_{N_f}, \vec{p}_{N_i}) e^{-i(p_{N_i} - p_{N_f}) \cdot (x_1 - x_2)}, \quad (92)$$

$$d(\vec{p}_{N_f}, \vec{p}_{N_i}) = \frac{1}{2} \sum_{spin} (\bar{u}(\vec{p}_{N_f}) \gamma_5 u(\vec{p}_{N_i}))^* \bar{u}(\vec{p}_{N_f}) \gamma_5 u(\vec{p}_{N_i}), \quad (93)$$

$$N = \frac{m_N^2}{(2\pi)^3 E_{N_f}}.$$

Taking a sum of the final spins and an average over the initial spin, we have

$$d(\vec{p}_{N_i}, \vec{p}_{N_f}) = (p_{N_i} \cdot p_{N_f} + m_N^2) / 2m_N^2, \quad (94)$$

and the correlation function

$$\begin{aligned} \Delta(x_1, x_2) &= \frac{1}{(2\pi)^3} \int \frac{d\vec{p}_{N_f}}{E_{N_f}} (m_N^2 + p_{N_i} \cdot p_{N_f}) e^{-i(p_{N_i} - p_{N_f}) \cdot (x_1 - x_2)} \\ &= \int d^4 p \theta(p^0) \text{Im} \left[\frac{1}{p^2 - m_N^2 + i\epsilon} \right] (m_N^2 + p_{N_i} \cdot p) e^{-i(p_{N_i} - p) \cdot (x_1 - x_2)} \\ &= e^{-ip_{N_i} \cdot (x_1 - x_2)} \int d^4 p \theta(p^0) \text{Im} \left[\frac{1}{p^2 - m_N^2 + i\epsilon} \right] (m_N^2 + p_{N_i} \cdot p) e^{ip \cdot (x_1 - x_2)}. \end{aligned} \quad (95)$$

The integral over \vec{X}_π in Eq. (90) gives

$$\int d\vec{X}_\pi \frac{1}{V_i} = 1. \quad (96)$$

3.2.1 light-cone singularity 1

Here the formula for a relativistic field

$$\begin{aligned} \int d^4q \theta(q^0) \delta(q^2 + \tilde{m}^2) e^{iq \cdot \delta x} &= (2\pi)^3 i \left[\frac{1}{4\pi} \delta(\lambda) \epsilon(\delta t) + f_{short} \right], \\ f_{short} &= -\frac{i\tilde{m}}{8\pi\sqrt{-\lambda}} \theta(-\lambda) \left\{ N_1 \left(\tilde{m}\sqrt{-\lambda} \right) - i\epsilon(\delta t) J_1 \left(\tilde{m}\sqrt{-\lambda} \right) \right\} \\ &\quad - \theta(\lambda) \frac{i\tilde{m}}{4\pi^2\sqrt{\lambda}} K_1 \left(\tilde{m}\sqrt{\lambda} \right), \quad \lambda = \delta x^2, \delta t = \delta x^0, \end{aligned} \quad (97)$$

where N_1 , J_1 , and K_1 are Bessel functions, is substituted into Eq. (95). Then $\Delta(x_1, x_2)$ is written as a sum of a singular term and other terms expressed by Bessel functions of $\lambda = (x_1 - x_2)^2$

$$\begin{aligned} \Delta(x_1, x_2) &= e^{-ip_{N_i} \cdot (x_1 - x_2)} \left(\frac{1}{4\pi} \delta(\lambda) \epsilon(t_1 - t_2) + \text{Bessel functions} \right) \\ &= e^{-i\bar{\phi}_{N_i}(\delta t)} \frac{1}{4\pi} \delta(\lambda) \epsilon(\delta t) + \text{Bessel functions}, \\ \bar{\phi}_{N_i} &= (E_{N_i} - |\vec{p}_{N_i}|) \delta t, \quad \delta t = t_1 - t_2. \end{aligned} \quad (98)$$

The first term in the right hand side of Eq. (97) is the most singular term and the second and third terms have singularity of the form $1/\lambda$ around $\lambda = 0$ and decrease as $e^{-\tilde{m}\sqrt{|\lambda|}}$ or oscillates as $e^{i\tilde{m}\sqrt{|\lambda|}}$. The singular functions and regular functions behave differently and are expressed in Fig. 2 for one space dimension. The singular functions are non-vanishing in narrow regions around the light cone and the regular functions have finite values in a small area around the origin. Since the light cone is extended in macroscopic area, the light-cone singularity makes the correlation function long-range. Thus the correlation function \tilde{I}_1 becomes long-range only along the light-cone region and decreases exponentially or oscillates rapidly in other directions. In the above equation, “Bessel functions” are oscillating or decreasing fast with λ and this property is sufficient to know a large T behavior of the probability. In latter sections, concrete forms of these functions are obtained and their properties are studied.

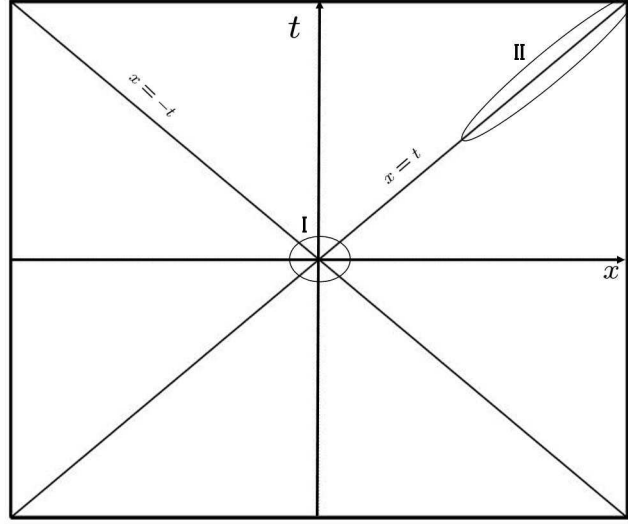


Fig. 2: The space time region of the correlation function is shown. The region I around the origin corresponds to short-range correlation where $t \sim x \sim 0$, while the region II corresponds to long-range correlation where $t^2 - x^2 \sim 0$ and both t and x can be macroscopic.

Eq. (98) is substituted into Eq. (91) and after a tedious calculation which will be presented in II, the probability is written in the form,

$$\begin{aligned} \tilde{P} &= \int_{T_{N_i}}^{T_\pi} dt_1 dt_2 (\sigma_\pi)^{\frac{3}{2}} \frac{\sigma_\pi}{2} \frac{1}{|\delta t|} \epsilon(\delta t) e^{i(\bar{\phi}_\pi(\delta t) - \bar{\phi}_{N_i}(\delta t))} + \text{Bessel functions}, \\ \bar{\phi}_\pi &= \omega_\pi \delta t, \quad \omega_\pi = (E_\pi - |\vec{p}_\pi|), \quad \delta t = t_1 - t_2, \end{aligned} \quad (99)$$

where we use the notation $t = x^0$ for the sake of simplicity. The probability derived from the first term of Eq. (99) is proportional to the following function of ωT expressed as

$$\begin{aligned} Tg(T, \omega) &= i \int_0^T dt_1 dt_2 \frac{e^{i\omega \delta t}}{|\delta t|} \epsilon(\delta t) = -2 \left(\text{Sin } x - \frac{1 - \cos x}{x} \right), \\ x &= \omega T, \quad \text{Sin } x = \int_0^x dt \frac{\sin t}{t}, \end{aligned} \quad (100)$$

which satisfies

$$\frac{\partial}{\partial T}g(T, \omega)|_{T=0} = -\omega, \quad (101)$$

$$g(\infty, \omega_\nu) = -\pi. \quad (102)$$

Subtracting the asymptotic value, we define $\tilde{g}(T, \omega_\nu)$

$$g(T, \omega) = \tilde{g}(T, \omega) + g(\infty, \omega), \quad (103)$$

which satisfies

$$\frac{d}{dx}\tilde{g}(x) = -4 \left(\frac{\sin(x/2)}{x} \right)^2, \quad (104)$$

and oscillates rapidly at large x . Averages become

$$\tilde{g}(x) = \frac{2}{x}. \quad (105)$$

Thus $\tilde{g}(T, \omega)$ behaves as

$$\tilde{g}(T, \omega) \approx \frac{T_0}{T}, \quad T_0 = \frac{2}{\omega}, \quad (106)$$

and is given in Fig. 3.

The probability derived from the second term of Eq. (99) is a constant. Thus the probability is expressed in the form

$$P = C'_0 \tilde{g}(\omega, T) + P_0, \\ \omega = (E_\pi - |\vec{p}_\pi|) - (E_{N_i} - |\vec{p}_{N_i}|), \quad T = T_\pi - T_{N_i}. \quad (107)$$

$\tilde{g}(\omega, T)$ is a function of ωT and is inversely proportional to ωT at a large ωT . Here ω is determined by the pion's mass and energy and the nucleon's mass and energy and is not very small. Hence, $\tilde{g}(\omega T)$ vanishes at a macroscopic T . The integrand of the probability in Eq. (99) oscillates rapidly in δt with a non-small angular velocity ω , and the probability becomes constant fast.

3.2.2 light-cone singularity 2

We should note that the integrand of Eq. (95) around a momentum region of $(p_{N_f} - p_{N_i})^2 = 0$ does not oscillate at $\lambda = 0$ and becomes real. Those waves

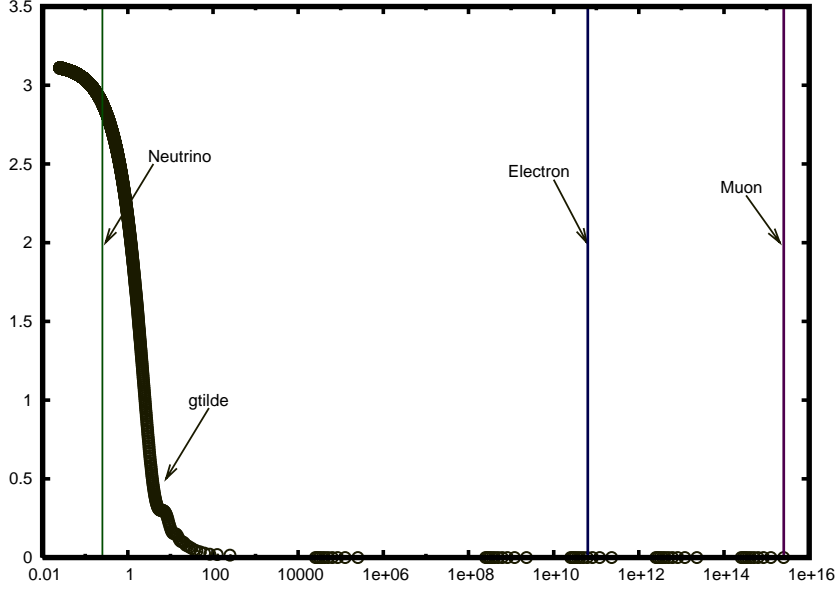


Fig. 3: The function $\tilde{g}(T, \omega)$ are given for various particles. The mass $m_\nu = 1$ [eV] is used for the neutrino in this Figure. Since the electron and muon are massive, $\tilde{g}(T, \omega)$ are negligibly small. The values for the pion and other hadrons are smaller than that of muon.

that are added constructively form the light-cone singularity of the form that does not accompany the oscillating function $e^{-i\phi_{N_i}(\delta t)}$. This function is extracted in the following way. We change the momentum from p to $q = p_{N_f} - p$, and have the expression with the new variable q

$$\Delta(x_1, x_2) = \int d^4q \theta(p_{N_i}^0 - q^0) \text{Im} \left[\frac{1}{(q - p_{N_i})^2 - m_N^2 + i\epsilon} \right] (2m_N^2 - p_{N_i} \cdot q) e^{-iq \cdot (x_1 - x_2)}. \quad (108)$$

Next the denominator is expanded in the form,

$$\begin{aligned}
\frac{1}{(q - p_{N_i})^2 - m_{N_f}^2 + i\epsilon} &= \frac{1}{q^2 + m_{N_i}^2 - 2q \cdot p_{N_i} - m_{N_f}^2 + i\epsilon} \\
&= \frac{1}{q^2 - 2q \cdot p_{N_i} + \delta m_N^2 + i\epsilon} \\
&= \sum_l \left(2q \cdot p_{N_i} \frac{\partial}{\partial \delta m_N^2} \right)^l \frac{1}{q^2 + \delta m_N^2 + i\epsilon}, \quad (109)
\end{aligned}$$

where a small difference between m_{N_i} and m_{N_f} which has been ignored so far is included here and $\delta m_N^2 = m_{N_i}^2 - m_{N_f}^2$ is the mass-squared difference between a proton and a neutron. We substitute this expression to $\Delta(x_1, x_2)$. From the integration of $q^0 < 0$ in Eq. (108), the light-cone singular term

$$\Delta(x_1, x_2) = \frac{1}{4\pi} \delta(\lambda) \epsilon(\delta t) + \text{Bessel functions}, \quad (110)$$

is derived from the lowest $l = 0$ term of Eq. (109). Eq. (110) is convenient to compute the large δt behavior of $\Delta(x_1, x_2)$. $\Delta(x_1, x_2)$ has also non-singular terms. This method of extracting the singularity is valid when the infinite series converges. The integrations of the series in the expansion of the denominator converge in the kinematical region,

$$2p_{N_i} \cdot p_\pi \leq \delta m_N^2. \quad (111)$$

Hence the present method is valid and the light-cone singularity exists in this narrow kinematical region. The convergence of the series will be studied in II in detail.

Eq. (110) is substituted into Eq. (91) and after tedious calculations, we have the probability in the form,

$$\begin{aligned}
\tilde{P} &= \int_{T_{N_i}}^{T_\pi} dt_1 dt_2 (\sigma_\pi)^{\frac{3}{2}} \frac{\sigma_\pi}{2} \frac{1}{|\delta t|} \epsilon(\delta t) e^{i\bar{\phi}_\pi(\delta t)} + \text{others}, \\
&= C_0(\sigma_\pi) \tilde{g}(\omega_\pi, T) + P_0. \quad (112)
\end{aligned}$$

In Eq. (112), P_0 is the normal term and the first term is the finite-size correction. The correction is the product of the universal function $\tilde{g}(\omega_\pi, T)$ that is independent of the wave packet and $C_0(\sigma_\pi)$.

Eq. (112) has almost the same form as Eq. (107) but a different angular velocity ω_π is used. ω_π is smaller than ω of Eq. (107). Hence this is more convenient to study the large T behavior than the former one since it has the slowest oscillation. At large $|\delta t|$ region, the frequency in time is given from

$$\bar{\phi}(\delta t) = (E_\pi - |\vec{p}_\pi|)\delta t = \frac{m_\pi^2}{2E_\pi}\delta t. \quad (113)$$

In the last calculation, the large momentum expansion $E(\vec{p}) = |\vec{p}| + \frac{m^2}{2|\vec{p}|}$ was made. The pion's mass m_π , 139 [MeV/c²], makes the angular velocity $\frac{m_\pi^2}{2E_\pi}$ small only in an extreme high-energy region. Thus a coherence length of a pion emitted from a nucleon is $l_0 = \frac{2cE_\pi}{m_\pi^2}$, which is microscopic due to the large mass, and the pion in the kinematical region, Eq. (111) can be observed inside this length. If the pion's mass m_π were 1 [eV/c²], then l_0 would have been macroscopic.

3.3 Pion from NN collisions

A pion produced in a nucleon collision shows the same behavior as a pion from an isolated nucleon of the previous subsection. Since a mass-shell nucleon lives the longest period of time compared to a virtual nucleon state, a pion emitted from a real state of nucleon has the longest correlation. This pion is expressed with external lines in Feynman diagram Fig. 4. We study this process hereafter in this section. Using a scattering amplitude of a proton with a target, $T(p_{N_f}, \dots, p_n; X_T)$, an amplitude of detecting one pion of a momentum \vec{p}_π at a position X^μ is

$$\int d^4x d\vec{k}_\pi w(\vec{p}_\pi - \vec{k}_\pi; X_\pi) e^{i(p_{N_f} + k_\pi - p_{N_i}) \cdot x} \bar{u}(p_{N_f}) \gamma_5 u(p_{N_i}) T(p_{N_i}, \dots, p_n; X_T), \quad (114)$$

where p_{N_i} and p_1, p_2, \dots are momenta of the initial and final state, and X_T is the position of the target, and x_1 is the coordinate where the pion in the external line attaches to the nucleon. We study a process expressed with a diagram of Fig. 4. Using a scattering amplitude, a correlation function is expressed in the form,

$$C(X_\pi, \vec{p}_\pi) = \int d\vec{p}_{N_i} P(X_\pi, \vec{p}_\pi; \vec{p}_{N_i}) |T(p_{N_i}, \dots, p_n; X_T)|^2, \\ w(\vec{p} - \vec{k}, X) = N'_0 e^{-\sigma(\vec{k} - \vec{p})^2 - ik \cdot X}, \quad (115)$$

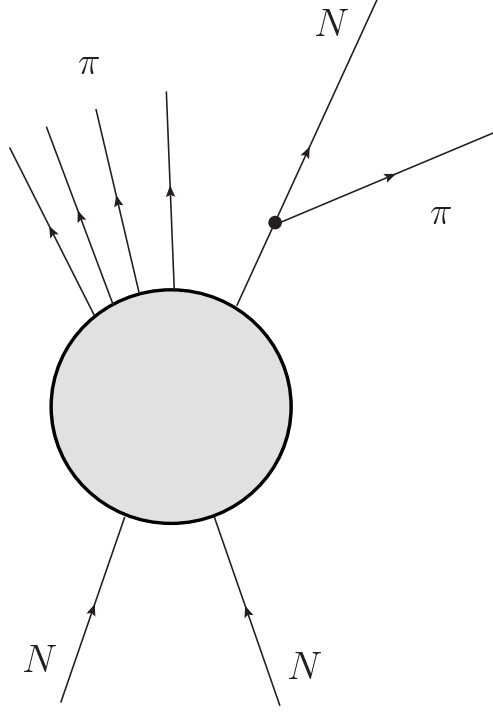


Fig. 4: A Feynman diagram of a pion production in NN scattering, where the pion has the longest correlation length. The pion emitted from the external nucleon has the longest correlation and is computed in the text.

where $P(X_\pi, p_\pi; p_{N_i})$ is the probability of a pion from a nucleon of the momentum p_{N_i} given in the previous subsection. Thus the behavior of $C(X_\pi, \vec{p}_\pi)$ is equivalent to that of $P(X_\pi, p_\pi; p_{N_i})$ and has a length l_0 of Eq. (13). The length $l_0 = \frac{c\hbar E_\pi}{m_\pi^2}$ is much longer than the de Broglie wave length, $\lambda = \frac{h}{|\vec{p}|}$, but is microscopic value in ordinary high energy experiments.

Thus the pion's coherence length l_0 is of microscopic size and experiments of detecting pions have no finite-size corrections. Pions are produced in small area at hadron collisions with certain probability and in the region $|\vec{X}_\pi - \vec{X}_T| \gg l_0$, they behave like particles.

4 Summary

The probability to detect a pion emitted from a nucleon is given in Eq. (112) and is decomposed to the normal term P_0 and the finite-size correction $C_0(\sigma)\tilde{g}(\omega T)$. The finite-size correction is expressed by a universal function $\tilde{g}(\omega T)$ in relativistic systems, and is stable under the change of parameters. The magnitude $C_0(\sigma)$ depends on σ and increases with σ . The finite-size correction is small for a pion and large for a light particle. From the position dependence of the correction, new physical quantities may be lead in light particles.

Thus scattering and decay processes of finite-time intervals are different from those of the infinite-time interval and have various unusual properties. A wave function started from an eigenstate of H_0 and evolves with an interaction Hamiltonian H_1 is a superposition of states of continuous spectrum of H_0 that includes the state of infinite momentum at a finite time. Despite the fact that the total energy defined by H is always conserved, the kinetic energy defined by the free part H_0 is not conserved at a finite-time interval due to a many-body interaction effect.

At the infinite time $t \rightarrow \infty$, only the states of $E_\beta = E_0$ couple to the wave function. Consequently the asymptotic values are determined by the states of conserving kinetic energy. At a finite time, on the other hand, various states of different kinetic energies couple. These states of continuous kinetic energy give a non-uniform behavior to many-body wave function and a diffraction phenomenon.

The amplitude and probability to detect particles at a finite distance reflect the unusual behavior of the wave function at a finite time. They satisfy boundary condition at a finite-time interval T , and are formulated with the $S[T]$, S-matrix at a finite T , expressed by wave packets. Wave packets are known for a long time to be a powerful tool for formulating scattering amplitudes in rigorous manners. But they were replaced with plane waves in practical calculations, because the finite-size corrections have been thought ignorable in high energy regions so far. We show in this paper that the wave packets are not only useful for formulating the scattering amplitude but also can be used for computing finite-size corrections of the probability to detect the particles in quantum mechanical processes. Particle's energy is given in a relativistic system as $E_\beta = \sqrt{\vec{p}^2 + m^2}$ and approaches $|\vec{p}|$ at the ultra-violet region. Hence the corresponding waves have almost the light velocity and cause the light-cone singularity to correlation functions. The light-cone

singularity helps to compute the finite size corrections at long range region.

The finite-size corrections to the probability to detect the pion produced in hadron reactions are studied as an example in I. We found that the coherence length for the wave phenomena to appear is small for the pion, and the correction is negligible in ordinary experiments.

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Appendix A Wave packet shape

Scattering matrix is expressed by wave packets in rigorous manner [14]. Wave packets are important, therefore, but they can be replaced with plane waves in practical situations in the asymptotic regions where the final state can be treated as free particles. In non-asymptotic regions where the final state retains wave nature, the wave packets are necessary to compute the position dependent quantities. The wave packets may be unfamiliar to some readers and we give a summary of wave packets.

A particle of a finite coherence length is described with a wave packet, which is extended in momentum and localized in position around the centers (\vec{p}, \vec{X}) and satisfies a corresponding wave equation. Although its precise shape may be unknown generally in real experiments, the physical quantities of universal properties that are independent from the details of wave packets are important.

First, the wave packets are localized in the momentum and position around their centers [23, 24, 25, 26]. The whole set of the wave packets becomes complete set [21],

$$\sum_{\vec{p}, \vec{X}} |\vec{p}, \vec{X}, \beta\rangle \langle \vec{p}, \vec{X}, \beta| = 1, \quad (116)$$

independent of the shape parameter β which is discussed later.

Second, we study the wave packets that preserve the discrete symmetries such as invariances under space and time inversions, which are quite natural, since the origin of the wave packet is the interactions of the particle with matters in detectors. The wave packet sizes are estimated in the next section under this consideration. We study, hence, the wave packets which are superpositions of the plane waves around the central momentum with a weight function that has the same property under these transformations,

$$\int d\vec{k} w(\vec{k}; \vec{p}) e^{i(Et - \vec{k} \cdot \vec{x})}, \quad (117)$$

where the momentum \vec{p} is the central value of the momentum. In the present work, \vec{k} is used for the integration variable and \vec{p} is used for the central value of momentum.

Since the time reversal invariance is satisfied in the quantum electrodynamics and weak interactions in the lowest order, the wave packet which is invariant under the time inversion is most important. Under the time inversion, the coordinate and momentum variables are transformed into

$$\vec{x} \rightarrow \vec{x}, \quad \vec{k} \rightarrow -\vec{k}, \quad \vec{p} \rightarrow -\vec{p}, \quad t \rightarrow -t, \quad (118)$$

and the plane wave is transformed to its complex conjugate,

$$e^{i(Et - \vec{k} \cdot \vec{x})} \rightarrow e^{-i(Et - \vec{k} \cdot \vec{x})} = (e^{i(Et - \vec{k} \cdot \vec{x})})^*. \quad (119)$$

So when the weight satisfies

$$w(\vec{k}; \vec{p}) \rightarrow w(-\vec{k}; -\vec{p}) = (w(\vec{k}; \vec{p}))^*, \quad (120)$$

the state described by the wave packet is the superposition of the waves which are transformed equivalently under the time inversion,

$$\int d\vec{k} w(\vec{k}; \vec{p}) e^{i(Et - \vec{k} \cdot \vec{x})} \rightarrow \int d\vec{k} w(-\vec{k}; -\vec{p}) (e^{i(Et - \vec{k} \cdot \vec{x})})^* = \left(\int d\vec{k} w(\vec{k}; \vec{p}) e^{i(Et - \vec{k} \cdot \vec{x})} \right)^*. \quad (121)$$

Next we study the space inversion. Although neutrino violates this symmetry, the wave packet may preserve. Under the space inversion, the coordinate and momentum variables are changed into

$$\vec{x} \rightarrow -\vec{x}, \quad \vec{k} \rightarrow -\vec{k}, \quad \vec{p} \rightarrow -\vec{p}, \quad (122)$$

and the plane wave is changed into

$$e^{i(Et - \vec{k} \cdot \vec{x})} \rightarrow e^{i(Et - \vec{k} \cdot \vec{x})}. \quad (123)$$

So when the weight satisfies

$$w(\vec{k}; \vec{p}) \rightarrow w(-\vec{k}; -\vec{p}) = w(\vec{k}; \vec{p}), \quad (124)$$

the state described by the wave packet is transformed in the following way,

$$\int d\vec{k} w(\vec{k}; \vec{p}) e^{i(Et - \vec{k} \cdot \vec{x})} \rightarrow \int d\vec{k} w(-\vec{k}; -\vec{p}) e^{i(Et - \vec{k} \cdot \vec{x})} = \int d\vec{k} w(\vec{k}; \vec{p}) e^{i(Et - \vec{k} \cdot \vec{x})}, \quad (125)$$

in the same way as the plane wave.

The invariance under the time reversal, Eq. (120), is a strong condition and leads the important result. We study wave packets that satisfy Eq. (120). The simplest form of satisfying this property is the Gaussian wave packet $|\vec{p}, \vec{X}, \beta_0\rangle$

$$|\vec{p}, \vec{X}, \beta_0\rangle = \frac{N}{(2\pi)^{\frac{3}{2}}} \int d\vec{k} e^{-\frac{\sigma}{2}(\vec{k} - \vec{p})^2} e^{i(E(\vec{k})(t-T) - \vec{k} \cdot (\vec{x} - \vec{X}))}, \quad (126)$$

where the parameter σ shows the size of the wave packet in the coordinate space and N is the normalization factor. We write a wave packet of a parameter β as $|\vec{p}, \vec{X}, \beta\rangle$. The Gaussian wave packet is the state of $\beta = \beta_0$ and non-Gaussian wave packets are the states of $\beta \neq \beta_0$. The calculations using

the Gaussian wave packet are presented in most places for the sake of simplicity, except the derivation of the large distance behavior of the neutrino wave function, which will be made in II.

The normal physical quantity of microscopic physics is obtained from ordinary scattering which has no dependence upon the distance or time interval between the initial and final states. This is because the size of the microscopic system is so small that the experimental apparatus is regarded infinite and the boundary conditions of ordinary scatterings which are defined at $t = \pm\infty$ of plane waves are suitable. Hence the amplitude and probability are defined using plane waves and the boundary conditions at $t = \pm\infty$ ensure the independence of the probability of the distance and particle's coherence length.

From Eq. (116), the total probability of the transition $A \rightarrow B + C$ is independent of the parameter β of C

$$\begin{aligned} & \sum_{\vec{p}_C, \vec{X}_C} |\langle C; \vec{p}_C, \vec{X}_C, \beta; B | T | A; \vec{p}_A, \vec{X}_A, T_A \rangle|^2 \\ &= \sum_{\vec{p}_C, \vec{X}_C} |\langle C; \vec{p}_C, \vec{X}_C, \beta_0; B | T | A; \vec{p}_A, \vec{X}_A, T_A \rangle|^2, \end{aligned} \quad (127)$$

which also agrees to that of the plane waves. The probability of the finite distance is defined by restricting the center positions \vec{X} inside a finite spatial region V . The completeness condition for the function in this region V is then

$$\sum_{\vec{p}, \vec{X} < V} |\vec{p}, \vec{X}, \beta_0\rangle \langle \vec{p}, \vec{X}, \beta_0| = \sum_{\vec{p}, \vec{X} < V} |\vec{p}, \vec{X}, \beta\rangle \langle \vec{p}, \vec{X}, \beta|, \quad (128)$$

and the probability satisfies

$$\begin{aligned} & \sum_{\vec{p}_C, \vec{X}_C < V} |\langle C; \vec{p}_C, \vec{X}_C, \beta; B | T | A; \vec{p}_A, \vec{X}_A, T_A \rangle|^2 \\ &= \sum_{\vec{p}_C, \vec{X}_C < V} |\langle C; \vec{p}_C, \vec{X}_C, \beta_0; B | T | A; \vec{p}_A, \vec{X}_A, T_A \rangle|^2, \end{aligned} \quad (129)$$

and is also independent from the β . We study the probability that is the average over a finite energy region V_p

$$\sum_{\vec{p}_C < V_p, \vec{X}_C < V} |\langle C; \vec{p}_C, \vec{X}_C, \beta; B | T | A; \vec{p}_A, \vec{X}_A, T_A \rangle|^2, \quad (130)$$

and extract the physical quantity. For the genuine physical quantity, the value should be determined uniquely. For the neutrino experiments, the energy uncertainty is of the order of 10 percent of the total neutrino energy. This magnitude is the same order as that of the minimum uncertainty if the neutrino energy is 1 [GeV]. So the minimum wave packet is applied directly in this case. The probability for a larger energy uncertainty is computed using the probability of the small energy uncertainty. The dependence of the neutrino probability on the distance will be shown to have the universal behavior.

The S-matrix at a finite-time interval T , $S[T]$, depends on the boundary condition at T , which is determined by the wave packet size σ of the final state. Hence the probability from $S[T]$ and finite-size correction are independent of parameter β but depends on σ . It is found in fact that the finite-size correction has the factorized form of $\tilde{g}(\omega T)$ and $C(\sigma)$. Only the latter depends on σ .

Appendix B Wave packet size

We estimate the wave packet size of a proton first and those of a pion next, following a method of our previous works [21, 22].

B-I Proton mean free path

A mean free path of a charged particle is determined by its scattering rate with atoms in matter caused by Coulomb interaction. An energy loss is also determined by the same cross section. Data on the energy loss are summarized well in particle data summary [7] and are used for the evaluation of the proton's mean free path.

The proton's energy loss rate at a momentum, 1 [GeV/c], for several metals such as Pb, Fe, and others are

$$-\frac{dE}{dx} = 1 - 2 \text{ [MeV g}^{-1}\text{cm}^2\text{]}, \quad (131)$$

hence we have the mean free path of the 1 [GeV/c] proton in the material of a density ρ ,

$$L_{\text{proton}} = \frac{E}{\frac{dE}{dx} \times \rho} = \frac{1 \text{ [GeV]}}{(1 - 2) \times 10 \text{ [MeV g}^{-1}\text{cm}^{-1}\text{]}} = 50 - 100 \text{ [cm]}. \quad (132)$$

At a lower energy of the order of $0.2 [\text{GeV}/c]$, the energy loss rate of the proton is about $10 [\text{MeVg}^{-1}\text{cm}^2]$ and the mean free path is

$$L_{\text{proton}} = 10 [\text{cm}]. \quad (133)$$

A wave which describes a proton maintains coherence in matter for a finite distance of the mean free path, hence this wave is approximately described by a wave packet that has a size of the mean free path. We use the mean free path for a wave packet size of the proton $\sqrt{\sigma}_{\text{proton}}$,

$$\sqrt{\sigma}_{\text{proton}} = L_{\text{proton}}. \quad (134)$$

When a proton of this size is emitted into the vacuum or to a dilute gas from matter, the wave keeps the same size. The proton that is moving freely has a constant size in vacuum or dilute gas. The size varies when the proton is accelerated. If the potential energy \mathcal{V} is added to the proton of momentum p_{before} , then the final value of the momentum becomes p_{after} and satisfies

$$\sqrt{p_{\text{before}}^2 + m^2} + \mathcal{V} = \sqrt{p_{\text{after}}^2 + m^2}. \quad (135)$$

From Eq. (135) variants of the momentum satisfy

$$\begin{aligned} v_{\text{before}} \times \delta p_{\text{before}} &= v_{\text{after}} \times \delta p_{\text{after}}, \\ v_{\text{before}} &= \frac{p_{\text{before}}}{\sqrt{p_{\text{before}}^2 + m^2}}, \quad v_{\text{after}} = \frac{p_{\text{after}}}{\sqrt{p_{\text{after}}^2 + m^2}}. \end{aligned} \quad (136)$$

Hence the size of a particle, $\sqrt{\sigma}_{\text{before}}$, which is proportional to the inverse of δp_{before} , becomes $\sqrt{\sigma}_{\text{after}}$ after the acceleration from a velocity v_{before} to a velocity v_{after} . The wave packet size is determined by the velocity ratio,

$$\sqrt{\sigma}_{\text{after}} = \sqrt{\sigma}_{\text{before}} \times \frac{v_{\text{after}}}{v_{\text{before}}}. \quad (137)$$

The velocity is bounded by the light velocity c , and a velocity ratio from $1 [\text{GeV}/c]$ to $10 [\text{GeV}/c]$ is about 1.2 and that from $0.2 [\text{GeV}/c]$ to $10 [\text{GeV}/c]$ is about 5. Hence the proton of $10 [\text{GeV}/c]$ regardless of the energy in matter has the mean free path

$$\sqrt{\sigma}_{\text{proton}} \approx 40 - 100 [\text{cm}], \quad (138)$$

in vacuum or a dilute gas.

B-II Pion wave packet

Wave packet size of pions which are produced by a proton collision with target nucleus is determined by the proton's initial size Eq. (138) and a target size 10^{-15} [m], which is negligibly small. A pion is produced while the proton wave packet passes through the small target by the strong interaction, hence this pion has a size in temporal direction of the proton wave packet. Hence the size of pion wave packet, $\sqrt{\sigma}_{\text{pion}}$, is given from that of the proton, $\sqrt{\sigma}_{\text{proton}}$, in the form

$$\frac{\sqrt{\sigma}_{\text{proton}}}{v_{\text{proton}}} = \frac{\sqrt{\sigma}_{\text{pion}}}{v_{\text{pion}}}, \quad \sqrt{\sigma}_{\text{pion}} = \frac{v_{\text{pion}}}{v_{\text{proton}}} \sqrt{\sigma}_{\text{proton}} \approx \sqrt{\sigma}_{\text{proton}}. \quad (139)$$

In relativistic energy regions, particles have the light velocity. Consequently from Eq. (138), pion's wave function of 1 [GeV/c] or larger momentum has the size

$$\sqrt{\sigma}_{\text{pion}} \approx 40 - 100 \text{ [cm]}. \quad (140)$$

We use this value of Eq. (140) as the size of the wave packet

$$\sqrt{\sigma}_{\pi} = \sqrt{\sigma}_{\text{pion}}, \quad (141)$$

in II.

In vacuum and dilute gas, pions of the size Eq.(141) propagate freely. From Eqs. (132), (138), (140), the proton and pion have the sizes of the order of 50 – 100 [cm].

Appendix C Neutrino on target: neutrino wave packet

A neutrino interacts in matter with a nucleon or an electron which are constituent particles in a bound atom and are expressed with wave functions of finite sizes. So a wave function expressing the final state of neutrino has a size of nucleus or atom, which is written with a wave packet. A size of wave packet for an observed neutrino, therefore, is not determined with a mean free path but with a size of the target in its detection process. They are either a size of a nucleons in a nucleus or that of an electron in an atom. Nucleus have sizes of the order of 10^{-15} [m] and electron's wave functions have

sizes of the order of 10^{-11} [m]. So neutrino wave packet is either 10^{-11} [m] or 10^{-15} [m].

Interactions of muon neutrinos in detectors are

$$\nu_\mu + e^- \rightarrow e^- + \nu_\mu, \quad (142)$$

$$\nu_\mu + e^- \rightarrow \mu^- + \nu_e, \quad (143)$$

$$\nu_\mu + A \rightarrow \mu^- + (A+1) + X, \quad (144)$$

$$\nu_\mu + A \rightarrow \nu_\mu + A + X, \quad (145)$$

hence the size of the neutrino wave packet $\sqrt{\sigma_\nu}$ in processes (142) and (143) is of the order of 10^{-11} , [m]

$$\sqrt{\sigma_\nu} = 10^{-11} \text{ [m]}, \quad (146)$$

and the neutrino wave packet $\sqrt{\sigma_\nu}$ in processes (144) and (145) is of the order of 10^{-15} [m]

$$\sqrt{\sigma_\nu} = 10^{-15} \text{ [m]}. \quad (147)$$

Interactions of electron neutrinos in detectors are

$$\nu_e + e^- \rightarrow e^- + \nu_e, \quad (148)$$

$$\nu_e + A \rightarrow e^- + (A+1) + X, \quad (149)$$

$$\nu_e + A \rightarrow e + A + X. \quad (150)$$

The neutrino wave packet $\sqrt{\sigma_\nu}$ in processes (148) is of the order of 10^{-11} [m], Eq. (146), and the neutrino wave packet $\sqrt{\sigma_\nu}$ in processes (149) and (150) is of the order of 10^{-15} [m], Eq. (147). They are treated in the same way as the neutrino from the pion decay.

From Eqs. (146) and (147), the neutrino have the wave packet sizes of the order of 10^{-11} [m] or 10^{-15} [m].

For various nucleus the sizes are estimated from the nucleus size, $A^{2/3}/m_\pi^2$ as

$$\sigma_\nu = \begin{cases} 5.2/m_\pi^2; C \\ 6.35/m_\pi^2; O \\ 14.3/m_\pi^2; Fe \\ 18.9/m_\pi^2; Pb. \end{cases} \quad (151)$$

The probability to detect neutrinos is expressed with the wave packet for the detecting particle, which is determined by the size of nucleus in the detector. In this respect, the neutrino wave packet of the present work is different from some previous works of wave packets that are connected with flavor neutrino oscillations [27, 28, 29, 30, 31, 32, 33, 34], where one particle properties of neutrino at productions are studied.

Appendix D Charged particles on target:wave packets

Charged particles are detected by their electromagnetic interactions with atoms in matters. The electromagnetic interactions are mediated by massless photons and the forces are long range and are much stronger than the weak interaction. Hence successive interactions with many atoms, which are correlated quantum mechanically each other in solid, give signals of the particles. Thus the wave packet sizes of the charged particles would be much larger than the size of an atom but the size of many atoms. It would be reasonable to assume that the size is semi-microscopic, some number of the order of one times 10^{-10} [m]. This size might agree to those that have been considered before in textbooks [23, 24, 25, 26]. Although the wave packet sizes of the charged particles are much larger than those of the neutrinos, the diffraction components have the magnitudes, $\tilde{g}(\omega_{charge}, T)$, which are extremely small due to their large masses, and vanish at the macroscopic distance. Consequently the finite-size corrections may be negligible for charged particles.

D-1 Dependence on wave packet size

It is known that the probability at $T = \infty$ does not depend on the wave packet size. The result of the present paper in fact shows that this probability is independent of the wave packet size. Now the finite-size correction is different and depends on the wave packet size. This peculiar property of the finite-size correction is one of the important results of the present paper. This correction has the universal properties, which were computed with $S[T]$ defined by the wave packets.